

# Lift force in odd compressible fluids

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We compute the response matrix for a tracer particle in a compressible fluid with odd viscosity living on a two-dimensional surface. Unlike the incompressible case, we find that an odd compressible fluid can produce an odd lift force on a tracer particle. Using a “shell localization” formalism, we provide analytic expressions for the drag and odd lift forces acting on the tracer particle in a steady state and also at finite frequency. Importantly, we find that the existence of an odd lift force in a steady state requires taking into account the non-conservation of the fluid mass density due to the coupling between the two-dimensional surface and the three-dimensional bulk fluid.

Odd viscosity is a transport coefficient in two dimensions breaking parity and time-reversal symmetry, which can occur in passive fluids subject to a background magnetic field [1, 2], as well as in active chiral systems [3, 4]. Breaking these symmetries results in the emergence of novel phenomena, and motivates the study of the physical behavior of odd materials.

The signatures of odd viscosity in fluids have been explored in various contexts (see e.g. [4–25]). The experimental realization of an active odd fluid in Ref. [17] showed that the strongest signatures of odd behaviour, such as edge flow or the rotation of asymmetric droplets, are found at interfaces. Inserting a tracer or probe particle in an odd fluid naturally introduces a boundary, making it an ideal candidate to probe the odd properties of a fluid, and has been the subject of several numerical and theoretical studies [7, 9, 19, 22, 26, 27]. In particular, due to the parity-breaking nature of odd viscosity, symmetry allows a fluid with a constant velocity at infinity not only to induce a drag force on a tracer particle, but also a lift force, orthogonal to the movement of the tracer.

Surprisingly, such a lift force is absent<sup>1</sup> in *incompress-*

*ible* odd fluids [9], and the motion of a tracer particle cannot be used to detect signatures of odd properties in these systems.

This brings us to a variant of the more-than-a-century-old question: *how much force does a tracer particle in a odd fluid experience?* Answering this question typically requires finding a smooth and regular solution for the velocity profile of the fluid flows satisfying appropriate boundary conditions near the tracer particle and far away from it. This problem, commonly known as the Stokes paradox (see Ref. [28] for a modern formulation and review), is notoriously difficult in two-dimensional fluids [29–32]. A standard way of circumventing this paradox is by means of a matched asymptotic expansion [33–37] in which the solution near the tracer particle is iteratively matched across a boundary layer to a solution far away from the particle. However, this approach is not easily adaptable to fluids with odd viscosity.

In this Letter, we show that a tracer particle in an odd *compressible* fluid experiences a lift force proportional to the odd viscosity coefficient, and that compressibility is a necessary condition for the existence of an odd lift force in two dimensions. We use a “shell localization” formalism [38–40] to compute analytically the drag and lift forces on a tracer particle in two different situations: a fluid in a steady-state configuration, and a fluid excited by an external force with finite frequency. Our formalism allows us to explore the dependence of these forces on the fluid compressibility and on the excitation frequency.

Crucially —and this point was overlooked in previous studies on this subject in which an instantaneous density relaxation was considered [19]— we show that a lift force only persists in a steady state in systems for which the density is not conserved. Non-conservation of density is generic in active systems as a consequence of birth and death processes, for instance in “Malthusian

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<sup>1</sup> A nonvanishing odd lift force, on a tracer in an incompressible fluid, assuming no-slip boundary conditions, was obtained in Ref. [7], but was contradicted in Ref. [9]. The discrepancy can be traced to the computation of the force on the probe, which in Ref. [7] used an incorrect pressure field.

flocks” [41, 42], cellular tissues [43], and in chemotactic systems [44]. Furthermore, absence of mass density conservation in two dimensions can arise from exchanges with a three-dimensional fluid bulk [45, 46]. This is for instance the case if the odd properties of the fluid stem from the activity of chiral particles, such as bacteria [47] or spermatozoa [48] that swim in a three-dimensional fluid and can accumulate at a surface.

As a further step, we also investigate the response of a probe excited periodically. At finite frequency, we show that an odd lift force can be measured in compressible fluids even if the mass density is conserved. This paves the way towards measurements of odd transport coefficients using frequency-dependent micro-rheology.

*Compressible odd fluid.*— We consider a thin layer of an odd compressible viscous fluid at the interface between two bulk (even) fluids, for instance water and air. For simplicity, we consider this layer to be flat and infinitely thin, such that the odd fluid can be described effectively as two-dimensional. The stress tensor associated with the mechanical properties of the odd fluid with velocity field  $v_i$  reads

$$\sigma_{ij} = 2\eta_s \partial_{\langle i} v_{j \rangle} + 2\eta_o \partial_{\langle i} v_{j \rangle} + (\eta_b \partial_k v_k - P) \delta_{ij} , \quad (1)$$

where  $i, j$  denote two-dimensional Cartesian coordinates and where summation over repeated indices is implied. For an arbitrary tensor  $A_{ij}$ , we have introduced the notation  $A_{\langle ij \rangle} = (A_{ij} + A_{ji})/2 - A_{kk} \delta_{ij}/2$  for its traceless symmetric part, such that  $\partial_{\langle i} v_{j \rangle}$  is the fluid shear rate. We have also introduced the odd tensor contraction  $A_{\{ij\}} = (\varepsilon_{ik} A_{kj} + \varepsilon_{ik} A_{jk} + \varepsilon_{jk} A_{ki} + \varepsilon_{jk} A_{ik})/4$ , where  $\varepsilon_{ij}$  denotes the fully antisymmetric tensor in two dimensions with  $\varepsilon_{12} = -\varepsilon_{21} = 1$ . Finally, we denote by  $\eta_{s,b,o}$  the shear, bulk and odd viscosities of the fluid, and by  $P$  its pressure field.

We consider an odd fluid that is weakly compressible with an equation of state of the form

$$P(\rho) = P_0 + \chi \frac{(\rho - \rho_0)}{\rho_0} , \quad (2)$$

where  $\chi^{-1}$  is the compressibility,  $\rho$  is the local mass density while  $P_0$  and  $\rho_0$  are the reference pressure and density, respectively.

The fluid velocity and mass density satisfy the following coupled differential equations

$$\begin{aligned} \partial_t \pi_i + v_k \partial_k \pi_i &= \eta_s \partial_k \partial_k v_i + \eta_b \partial_i \partial_k v_k \\ &+ \eta_o \varepsilon_{ij} \partial_k \partial_k v_j - \partial_i P - \frac{1}{\tau} \pi_i + f_i , \end{aligned} \quad (3a)$$

$$\partial_t \rho + \partial_k (\rho v_k) = -\frac{1}{\kappa} (\rho - \rho_0) , \quad (3b)$$

where  $\pi_i = \rho v_i$  is the momentum density. Eq. (3a) represents the momentum balance equation, which includes a relaxation process with timescale  $\tau$  to account for friction between the two-dimensional fluid and the

three-dimensional bulk, as well as a force density  $f_i$  acting on the fluid. Similarly, Eq. (3b) is the mass balance equation, which includes a mass exchange process with timescale  $\kappa$  to account for particle exchange with the bulk of the fluid [46], or for birth and death processes. To simplify the system of coupled nonlinear differential equations (3), we linearize it to first order in  $v_i$  and  $\delta\rho = \rho - \rho_0$  near a vanishing velocity and homogeneous reference state. The balance equations take the form

$$\begin{aligned} \rho_0 \partial_t v_i &= \eta_s \partial_k \partial_k v_i + \eta_b \partial_i \partial_k v_k \\ &+ \eta_o \varepsilon_{ij} \partial_k \partial_k v_j - \partial_i P - \frac{\rho_0}{\tau} v_i + f_i , \end{aligned} \quad (4a)$$

$$\partial_t \delta\rho + \rho_0 \partial_k v_k = -\frac{1}{\kappa} \delta\rho . \quad (4b)$$

We will use these equations to compute the response of a probe to an external force in an odd fluid.

*Shell localization.*— Having defined the equations of motion, we move to Fourier space with the convention

$$g(t, x_i) = \frac{1}{(2\pi)^3} \int d\omega d^2 k g(\omega, k_i) e^{-i\omega t + ik_j x_j} , \quad (5)$$

for some function  $g(t, x_i)$  so that Eq. (4) can be written in matrix form  $\mathcal{M}_{ij} v_j = f_i$  with  $\mathcal{M}_{ij}$  given by

$$\begin{aligned} \mathcal{M}_{ij} &= \hat{k}_i \hat{k}_j \left[ \frac{\rho_0}{\tau} - i\omega \rho_0 + \left( \eta_s + \eta_b + \frac{\chi \kappa}{1 - i\omega \kappa} \right) k^2 \right] \\ &+ (\delta_{ij} - \hat{k}_i \hat{k}_j) \left[ \frac{\rho_0}{\tau} - i\omega \rho_0 + \eta_s k^2 \right] + \varepsilon_{ij} \eta_o k^2 , \end{aligned} \quad (6)$$

where  $k = \sqrt{k_i k_i}$  and  $\hat{k}_i = k_i/k$ . This matrix can be inverted as

$$v_i(k_i, \omega) = (\mathcal{M}^{-1})_{ij}(k_i, \omega) f_j(k_i, \omega) , \quad (7)$$

to yield the velocity induced by a force distribution. Specifically, we consider the force applied on a tracer particle, which is a rigid disk of radius  $a$  located at the origin. Due to the rotational symmetry of the disk, we can decompose the force density as  $f_j(k_i, \omega) = F(k) \mathcal{F}_j(\omega)$ .

The shell localization method consists in considering that the force density is located in real space according to [38, 39, 49]:

$$F(x) = \frac{1}{2\pi a} \delta(|x| - a) . \quad (8)$$

Eq. (8) enforces the force density exerted by the disk on the fluid to be uniformly distributed along the entire edge of the disk. The disk is coupled to the fluid through a no-slip boundary condition, which equates the velocity of the tracer particle to the fluid velocity at the edge of the tracer particle. Fourier transforming Eq. (8) yields  $F(k) = J_0(ak)$  with  $J_n(z)$  the  $n^{\text{th}}$  Bessel function of the first kind. The velocity of the disk located at  $|x| = 0$  is then directly given by the inverse Fourier transform at the origin

$$v_i(|x| = 0, \omega) = (\mathbb{M}^{-1})_{ij}(\omega) \mathcal{F}_j(\omega) , \quad (9)$$

where the “response matrix” is

$$(\mathbb{M}^{-1})_{ij}(\omega) = \int_0^{2\pi} d\theta \int_0^\infty dk k F(k) (\mathcal{M}^{-1})_{ij}(k_i, \omega). \quad (10)$$

The response matrix  $(\mathbb{M}^{-1})_{ij}(\omega)$  encodes the velocity of a rigid probe immersed in an odd fluid as a function of the applied (frequency-dependent) force  $\mathcal{F}_j(\omega)$ . Using the disk radius  $a$  we can introduce the dimensionless coefficients

$$\begin{aligned} z &= ak, \quad \tilde{\omega} = \omega a^2 \rho_0 / \eta_s, \quad \tilde{\eta}_o = \eta_o / \eta_s, \quad \tilde{\eta}_b = \eta_b / \eta_s \\ \tilde{\tau} &= \tau \eta_s / (\rho_0 a^2), \quad \tilde{\chi} = \chi \rho_0 a^2 / \eta_s^2, \quad \tilde{\kappa} = \kappa \eta_s / (\rho_0 a^2), \end{aligned} \quad (11)$$

so that Eq. (10) turns into

$$\begin{aligned} \mathcal{M}_{ij} &= \frac{\eta_s}{a^2} \left\{ \hat{z}_i \hat{z}_j \left[ \frac{1}{\tilde{\tau}} - i\tilde{\omega} + \left( 1 + \tilde{\eta}_b + \frac{\tilde{\chi} \tilde{\kappa}}{1 - i\tilde{\omega} \tilde{\kappa}} \right) z^2 \right] \right. \\ &\quad \left. + (\delta_{ij} - \hat{z}_i \hat{z}_j) \left[ \frac{1}{\tilde{\tau}} - i\tilde{\omega} + z^2 \right] + \varepsilon_{ij} \tilde{\eta}_o z^2 \right\}. \end{aligned} \quad (12)$$

Before considering the most general case of a compressible fluid, where a lift force can arise, we first discuss the limiting case of an incompressible odd fluid. This corresponds to the limit  $\tilde{\chi} \rightarrow \infty$ , for which the matrix  $\mathcal{M}^{-1}$  reads

$$\lim_{\tilde{\chi} \rightarrow \infty} (\mathcal{M}^{-1})_{ij}(z_k, \tilde{\omega}) = \frac{a^2}{\eta_s} \frac{\delta_{ij} - \hat{z}_i \hat{z}_j}{z^2 + \tilde{\tau}^{-1} - i\tilde{\omega}}. \quad (13)$$

It may be observed that this matrix is transverse to the wave-vector, indicating the absence of an odd lift force as expected for an incompressible odd fluid [9]. In addition, the odd viscosity transport coefficient is absent, indicating that the response of the tracer particle in the case of an odd incompressible fluid is identical to the response in the case of an even incompressible fluid. In App. A, we verify that in the incompressible case the shell localization gives a drag force that is consistent with results found by explicitly solving the boundary value problem in two instances. Specifically, we recover the result for two-dimensional oscillatory drag [50, 51] as well as the result for the drag force found in the Saffman-Delbrück model [52, 53], provided we appropriately match the relaxation time to the coefficients of this model [54].

*Odd lift force.*— We now address the general case of a compressible fluid. In this setting, the response matrix can be written as

$$(\mathbb{M}^{-1})_{ij}(\omega) = \frac{1}{\eta_s} (M_d \delta_{ij} - M_l \varepsilon_{ij}), \quad (14)$$

where  $M_d$  and  $M_l$  are respectively the dimensionless response functions for drag force, and for lift force, specific to compressible odd fluids.

*Steady-state odd lift force.*— We first consider the steady-state case  $\tilde{\omega} \rightarrow 0$  with a non-vanishing relaxation rate

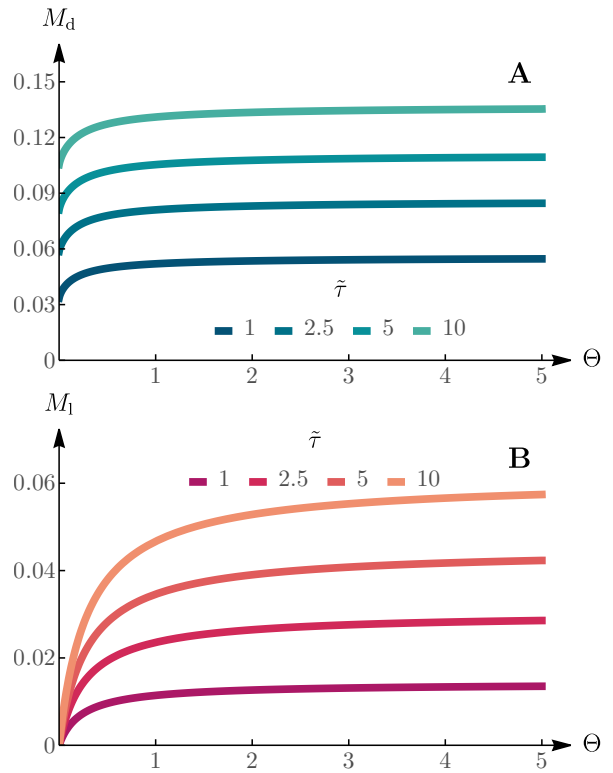


FIG. 1. Steady-state drag coefficient  $M_d$  (A) and lift coefficient  $M_l$  (B) as a function of the dimensionless inverse compressibility  $\Theta = (\tilde{\chi} \tilde{\kappa})^{-1}$  for different values of the relaxation time  $\tilde{\tau}$  and for  $\tilde{\eta}_b = \tilde{\eta}_o = 1$ .

$\tilde{\tau}^{-1} \neq 0$ . The drag and lift are obtained by computing the momentum integrals

$$M_d = \frac{1}{4\pi} \int dz J_0(z) \frac{D(z)}{Q(z)}, \quad (15a)$$

$$M_l = \frac{1}{2\pi} \int dz J_0(z) \frac{L(z)}{Q(z)}, \quad (15b)$$

where we have defined:

$$Q(z) = \tilde{\tau}^2 z^4 (\tilde{\eta}_b + \tilde{\eta}_o^2 + \Theta^{-1} + 1) + \tilde{\tau} z^2 (\tilde{\eta}_b + \Theta^{-1} + 2) + 1, \quad (16a)$$

$$D(z) = \tilde{\tau} z (\tilde{\tau} z^2 (\tilde{\eta}_b + \Theta^{-1} + 2) + 2), \quad L(z) = \tilde{\eta}_o \tilde{\tau}^2 z^3, \quad (16b)$$

and where  $\Theta^{-1} = \tilde{\kappa} \tilde{\chi}$ . As advertised in the introduction, we note that the odd lift force vanishes for  $\tilde{\eta}_o \rightarrow 0$ , which is expected as it can only be induced by a parity-odd coefficient. Furthermore,  $M_l$  is only non-vanishing when  $\tilde{\kappa}^{-1}$  is non-vanishing, since in the steady case the limit  $\tilde{\kappa} \rightarrow \infty$  is equivalent to the incompressible limit for which was shown in Eq. (13) that the lift force vanishes. This means that in a steady state there can only be lift forces when density is not conserved, for instance if exchanges between the surface and three-dimensional fluid, parameterized by the relaxation time  $\kappa$ , take place.

As we detail in App. B, the momentum integrals can be computed analytically using residues but their expression can become lengthy. For the purpose of clarity, we consider a series expansion in powers of the odd viscosity  $\tilde{\eta}_o$  and keep the first non-vanishing contribution. Specifically, we find

$$M_d = \frac{K_0(\tilde{\tau}^{-1/2}) + K_0[(\Xi\tilde{\tau})^{-1/2}]/\Xi}{4\pi} + \mathcal{O}(\tilde{\eta}_o^2), \quad (17a)$$

$$M_l = \frac{\tilde{\eta}_o [K_0(\tilde{\tau}^{-1/2}) - K_0[(\Xi\tilde{\tau})^{-1/2}]/\Xi]}{2\pi(\Xi - 1)} + \mathcal{O}(\tilde{\eta}_o^2), \quad (17b)$$

with  $\Xi = 1 + \tilde{\eta}_b + \Theta^{-1}$  and where  $K_n(x)$  is the  $n^{\text{th}}$  modified Bessel function of the second kind. In the incompressible fluid limit or for a compressible fluid without mass density relaxation ( $\Theta \rightarrow 0$ ), we find  $M_l = 0$  and  $M_d = K_0(\tilde{\tau}^{-1/2})/(4\pi)$ .

We now evaluate  $M_d$  and  $M_l$  from Eq. (15) as a function of  $\Theta$  and provide the result in Fig. 1. We take  $\tilde{\eta}_b = \tilde{\eta}_o = 1$  for the dimensionless viscosities. We observe in Fig. 1 A that the drag force is significantly affected by the momentum relaxation time  $\tilde{\tau}$  but only weakly depends on the compressibility parameter  $\Theta$ . On the other hand, Fig. 1 B shows the crucial role of the compressibility in the magnitude of the lift force, which vanishes in the incompressible limit  $\Theta \rightarrow 0$ .

We also consider the limit  $\Theta \rightarrow \infty$  which corresponds to an infinitely compressible fluid ( $\tilde{\chi} = 0$ ), or to a fluid with an instantaneous density relaxation ( $\kappa = 0$ ). In this limit, any deviation from the reference density  $\rho_0$  is instantly relaxed to the bulk, such that pressure is constant and plays no role in the response matrix. Our equations reduce in this case to the ones considered in Ref. [19] where numerical expressions for the response function are computed.

Lastly, we note that the odd lift coefficient  $M_l$  can become negative for small values of  $\tilde{\tau}$  and large values of  $\Theta$ . However, this regime in parameter space for which  $\tau \ll 1$  is precisely the regime in which momentum relaxation dominates and the system given by Eqs. (3) no longer provides an accurate description of two-dimensional fluid flows.

*Frequency-dependent lift force.*— We now consider the system in the absence of relaxation processes ( $\tilde{\tau}^{-1} \rightarrow 0$  and  $\tilde{\kappa}^{-1} \rightarrow 0$ ) to focus on the frequency-dependent response of the tracer. In Fig. 2, we display the real and imaginary parts of the drag coefficient  $M_d(\tilde{\omega})$  and odd lift coefficient  $M_l(\tilde{\omega})$  as a function of the dimensionless frequency  $\tilde{\omega}$  and for different values of the inverse compressibility  $\tilde{\chi}^{-1}$ . The drag coefficient  $M_d(\tilde{\omega})$  diverges as  $\tilde{\omega} \rightarrow 0$ , which is a signature of the Stokes paradox, see Figs. 2 A and B.

On the other hand, the lift coefficient  $M_l(\tilde{\omega})$  vanishes at steady state, see Figs. 2 C and D. At finite excitation frequency and compressibility, a nonvanishing odd response can be measured. Note that both the drag and

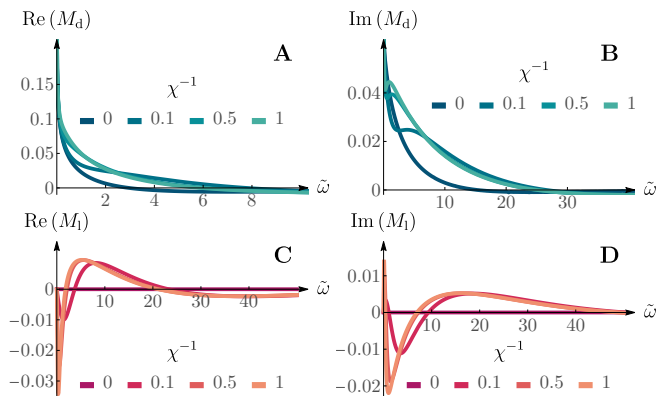


FIG. 2. Real (A,C) and imaginary parts (B,D) of the complex drag and lift coefficients  $M_{d,l}$  as a function of the dimensionless frequency  $\tilde{\omega}$  for different values of the inverse compressibility  $\tilde{\chi}$  and for  $\tilde{\eta}_b = \tilde{\eta}_o = 1$ .

lift responses vanish at large frequencies, as expected for a fluid.

Additionally, a simple analytic expression for the drag and odd lift coefficient  $M_{d,l}$  can be obtained by expanding Eq. (10) in the absence of relaxation processes ( $\tilde{\tau}^{-1} \rightarrow 0$  and  $\tilde{\kappa}^{-1} \rightarrow 0$ ) and at leading order in the inverse compressibility  $\tilde{\chi}^{-1}$ . One obtains

$$M_d = \frac{1}{4\pi} K_0 \left( \sqrt{\tilde{\omega}/i} \right) + \mathcal{O}(\tilde{\chi}^{-1}), \quad (18a)$$

$$M_l = \frac{-i\tilde{\omega}\tilde{\eta}_o}{2\pi\tilde{\chi}} K_0 \left( \sqrt{\tilde{\omega}/i} \right) + \mathcal{O}(\tilde{\chi}^{-2}). \quad (18b)$$

The drag and lift coefficients have a completely different behavior in the limit of small frequencies. Indeed, we have the expansion<sup>2</sup>

$$M_d = -\frac{1}{8\pi} \left( \log \frac{\tilde{\omega}}{4} + 2\gamma_{\text{EM}} - \frac{i\pi}{2} \right) + \mathcal{O}(\tilde{\chi}^{-1}, \tilde{\omega}), \quad (19a)$$

$$M_l = \frac{i\tilde{\omega}\tilde{\eta}_o}{4\pi\tilde{\chi}} \left( \log \frac{\tilde{\omega}}{4} + 2\gamma_{\text{EM}} - \frac{i\pi}{2} \right) + \mathcal{O}(\tilde{\chi}^{-2}, \tilde{\omega}^2), \quad (19b)$$

which shows a  $\log \tilde{\omega}$  divergence of the drag, as expected from the Stokes paradox, while the odd lift coefficient vanishes as  $\tilde{\omega} \log \tilde{\omega}$ . This difference in the small  $\tilde{\omega}$  behavior is clearly visible in Fig. 2.

*Discussion.*— In this Letter we obtained analytical expressions for the drag and lift coefficients of a disk in a two-dimensional odd compressible fluid. We used a shell localization approach [38, 39] to study the probe response both at steady-state and at finite frequency. In the incompressible limit, we confirmed the absence of odd effects on the tracer with no-slip boundary conditions [9].

<sup>2</sup> Note that because the drag coefficient diverges in the limit  $\tilde{\omega} \rightarrow 0$ , the expansion in a series of  $\tilde{\omega}$  must be performed after computing the momentum integral over  $z$ .

Surprisingly, we found that an odd lift force can only persist on the probe at steady state provided the density of the two-dimensional fluid is not conserved<sup>3</sup>. It would be interesting to see whether an “effective boundary condition” [56] accounting for a small finite compressibility can be used to capture the odd lift force on the probe while using an incompressible model in the bulk. Finally, when the tracer is excited at finite frequency  $\omega$ , we found that an odd lift response exists at finite frequency, and vanishes as  $\omega \log(\omega)$  in the limit of small frequency. For comparison, the drag diverges in the same limit as  $\log(\omega)$ , a signature of Stokes paradox [50, 51]. These results suggest that active micro-rheology could be used to measure the properties of odd viscoelastic materials.

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### Appendix A: Incompressible drag force

In this Appendix we explicitly compute the response matrix in two simple incompressible scenarios. In the absence of relaxation ( $\tilde{\tau}^{-1} \rightarrow 0$ ), the incompressible response matrix is given by

$$\begin{aligned} (\mathbb{M}^{-1})_{ij}(\tilde{\omega}) &= \frac{\delta_{ij}}{4\pi\eta_s} \int dz \frac{zJ_0(z)}{z^2 - i\tilde{\omega}} \\ &= -\frac{\delta_{ij}}{8\pi\eta_s} \left[ \log\left(\frac{\tilde{\omega}}{4}\right) + 2\gamma_{\text{EM}} - i\frac{\pi}{2} \right] + \mathcal{O}(\tilde{\omega}), \end{aligned} \quad (\text{A1})$$

where  $\gamma_{\text{EM}}$  is the Euler-Mascheroni constant. We see that this result is divergent in the steady-state limit ( $\tilde{\omega} \rightarrow 0$ ), which is a signature of the Stokes paradox [28]. Note that the shell localization result given in Eq. (A1) matches the drag force that one would obtain from solving explicitly

the Stokes equation with no-slip boundary conditions [50, 51].

A second scenario is the steady-steady case  $\tilde{\omega} \rightarrow 0$  with a finite relaxation rate  $\tilde{\tau}^{-1} \neq 0$ . The incompressible response matrix now reads

$$\begin{aligned} (\mathbb{M}^{-1})_{ij}(0) &= \frac{\delta_{ij}}{4\pi\eta_s} \int dz \frac{zJ_0(z)}{z^2 + 1/\tilde{\tau}} \\ &= \frac{\delta_{ij}}{4\pi\eta_s} \left[ \log\left(2\sqrt{\tilde{\tau}}\right) - \gamma_{\text{EM}} \right] + \mathcal{O}(\tilde{\tau}^{-1}). \end{aligned} \quad (\text{A2})$$

In this case, the response matrix is non-divergent thanks to the momentum relaxation circumventing the Stokes paradox [54, 57–59]. The result in Eq. (A2) can be compared to the result from works of Saffman and Delbrück [52, 53], if one matches the relaxation  $\tilde{\tau}$  as [54]

$$\tilde{\tau} = \left( \frac{\eta_s}{2a\eta'_s} \right)^2, \quad (\text{A3})$$

where  $\eta'_s$  is the shear viscosity of the surrounding bulk fluid that is tied to the substrate in Refs. [52, 53]<sup>4</sup>. We thus find that in these two instances, the shell localization approach yields the same results as in previous works where the fluid velocity profile is computed over the entire two-dimensional surface [52, 53].

### Appendix B: Analytical computation of the response matrix

In this Appendix, we show how the integrals performed throughout this Letter can be performed using the method of residues. For a compressible fluid as described in the main text, the response coefficients are obtained by performing momentum integrals that take the form

$$I[R] = \int_0^\infty dz zR(z)J_0(z), \quad (\text{B1})$$

where  $R(z) = A(z)/B(z)$  is an even function of  $z$ , and where  $A$  and  $B$  are polynomials in  $z$ . We call  $z_n$  the  $n^{\text{th}}$  root of  $B(z)$ , such that  $B(z_n) = 0$ . Following Ref. [60], the integral  $I[R]$  can be computed analytically in terms of the Hankel functions of the first kind  $H_\nu^{(1)}$  and the Bessel functions of the second kind  $Y_\nu$ . It reads:

$$\begin{aligned} I[R] &= i\pi \sum_{z_n \in \mathbb{C}^+ \setminus \mathbb{R}} \text{Res} \left( R(z)H_0^{(1)}(z), z_n \right) \\ &\quad - \pi \sum_{z_n \in \mathbb{R}^+} \text{Res} \left( R(z)Y_0(z), z_n \right), \end{aligned} \quad (\text{B2})$$

<sup>3</sup> Note that in Ref. [19], the odd lift force was computed in the limit of a vanishing density relaxation time ( $\kappa \rightarrow 0$ ) using the reciprocal theorem [55]. However, this theorem relies on the index exchange symmetry  $\eta_{ijkl} = \eta_{klij}$  of the viscosity tensor, which does not hold for an odd fluid.

<sup>4</sup> Note in Refs. [52, 53] the shear viscosity in the substrate  $\eta_s^{(SD)}$  is three-dimensional and therefore it has different units from the  $\eta_s$  appearing in this letter. In Eq. (A3) the two viscosities are related by taking  $\eta_s^{(SD)} \rightarrow \eta_s/h$ , with  $h$  being the height of the substrate.

where the first sum is over the roots of  $B(z)$  whose imaginary part is strictly positive, and the second one is over the positive real roots of  $B(z)$ . We denote by  $\text{Res}(f(z), z_n)$  the residue of  $f$  at point  $z_n$ .

As an illustration, we consider the oscillatory incompressible case in the absence of relaxation ( $\tilde{\tau}^{-1} \rightarrow 0$ ), for which one has

$$R(z) = \frac{1}{z^2 - i\tilde{\omega}}, \quad (\text{B3})$$

and thus for which  $A(z) = 1$  and  $B(z) = z^2 - i\tilde{\omega}$  with the

roots  $z_{1,2} = \pm\sqrt{i\tilde{\omega}}$ . In this case, only the first term in the right-hand side of Eq. (B2) contributes, and since the Hankel function  $H_0^{(1)}$  has no pole in  $z_1 = \sqrt{i\tilde{\omega}}$ , it yields

$$I[(z^2 - i\tilde{\omega})^{-1}] = \frac{i\pi}{2} H_0^{(1)}(\sqrt{i\tilde{\omega}}) = K_0(-i\sqrt{i\tilde{\omega}}), \quad (\text{B4})$$

where  $K_\nu(x)$  is the  $\nu^{\text{th}}$  modified Bessel function of the second kind. An expansion of Eq. (B4) in series of  $\tilde{\omega}$  yields the result given in Eq. (A1).

The same procedure can be applied for the compressible case, and was used to obtain Eqs. (17) and (18) of the main text.

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